BLOOMSBURY REVELATIONS

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INTENSIVE SCIENCE AND VIRTUAL PHILOSOPHY

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Chapter 1

The Mathematics of the Virtual: Manifolds, Vector Fields and Transformation Groups

Of all the concepts which populate the work of Gilles Deleuze there is one that stands out for its longevity: the concept of *multiplicity.* This concept makes its appearance in his early books and remains one of central importance, with almost unchanged meaning and function, until his final work.[1] Its formal definition is highly technical, including elements from several different branches of mathematics: differential geometry, group theory and dynamical systems theory. In this chapter I will discuss the technical background needed to define this important concept but some preliminary informal remarks will prove helpful in setting the stage for the formal discussion. In the first place, one may ask what role the concept of a multiplicity is supposed to play and the answer would be a replacement for the much older philosophical concept of an *essence.* The essence of a thing is that which explains its *identity,* that is, those fundamental traits without which an object would not be what it is. If such an essence is shared by many objects, then possession of a common essence would also explain the fact that these objects *resemble* each other and, indeed, that they form a distinct *natural kind* of things.

Let's take one of the most traditional illustrations of an essence. When one asks what makes someone a member of the human species the answer may be, for example, being a "rational animal". The exact definition of the human essence is not what is at issue here (if rationality and animality are not considered to be essential human properties some other set will do). The important point is that there be some set of defining characteristics, and that this set explain both the identity of the human species and the fact that particular members of the species resemble each other. In a Deleuzian ontology, on the other hand, a species (or any other natural kind) is not defined by its essential traits but rather by the *morphogenetic process* that gave rise to it. Rather than representing timeless categories, species are historically constituted entities, the resemblance of their members explained by having undergone common processes of natural selection, and the enduring identity of the species itself guaranteed by the fact that it has become reproductively isolated from other species. In short, while an essentialist account of species is basically static, a morphogenetic account is inherently dynamic. And while an essentialist account may rely on factors that transcend the realm of matter and energy (eternal archetypes, for instance), a morphogenetic account gets rid of all *transcendent* factors using exclusively form-generating resources which are *immanent* to the material world.

Animal and plant species are not, of course, the only natural kinds traditionally defined by essences. Many other natural kinds, the chemical elements or the set of elementary particles, for example, are also typically so defined. In each of these cases we would need to replace timeless categories by historical processes. Yet, even if successful this replacement would take us only half-way towards our goal. The reason is that even if the details of a given process account for the resemblance among its products, the similarities which make us classify them as members of the same kind, there may be *similarities of process* which still demand an explanation. And when accounting for these common features we may be tempted to reintroduce essences through the back door. These would not be essences of objects or kinds of objects, but essences of processes, yet essences nevertheless. It is in order to break this vicious circle that multiplicities are introduced. And it is because of the tenacity of this circle that the concept of multiplicity must be so carefully constructed, justifying each step in the construction by the way it avoids the pitfalls of essentialism. To anticipate the conclusion I will reach after a long and technical definitional journey: multiplicities specify *the structure of spaces of possibilities,* spaces which, in turn, explain the regularities exhibited by morphogenetic processes. I will begin by defining an appropriate notion of "space", a notion which must not be purely geometrical but also capable of being linked to questions of process.

The term "multiplicity" is closely related to that of "manifold", a term which designates a geometrical space with certain characteristic properties. To grasp what is special about manifolds (and what resources this concept can offer to avoid essentialism) it will be useful to give a brief account of its historical origins. Although the use of geometrical procedures for the solution of problems is an ancient practice inherited from the Greeks, the extensive use of curves and trajectories in the formulation of a variety of physical problems from the sixteenth century on made it necessary to develop new problem-solving resources. With this in mind, René; Descartes and Pierre de Fermat invented the now familiar method of embedding curves into a two-dimensional space on which arbitrary axes could be fixed. Once so embedded, the fixed axes allowed the assignment of a pair of numbers, or coordinates, to every point of the curve, so that the geometric relations between points could now be expressed as relations between numbers, a task for which the newly developed algebra was perfectly suited. This translation scheme, in short, allowed the combinatorial resources of algebra to be brought to bear on the solution of geometrical problems.

The term "manifold" does not belong to the analytical geometry of Descartes and Fermat, but to the *differential geometry* of Friedrich Gauss and Bernhard Riemann, but the basic idea was the same: tapping into a new reservoir of problem-solving resources, the reservoir in this case being the differential and integral calculus. In its original application the calculus was used to solve problems involving relations between the changes of two or more quantities. In particular, if these relations were expressed as a *rate of change* of one quantity relative to another, the calculus allowed finding the instantaneous value for that rate. For example, if the changing quantities were spatial position and time, one could find instantaneous values for the rate of change of one relative to the other, that is, for velocity. Using this idea as a resource in geometry involved the realization that a geometrical object, a curved line or surface, for instance, could also be characterized by the rate at which some of its properties changed, for example, the rate at which *its curvature* changed between different points. Using the tools of the calculus mathematicians could now find "instantaneous" values for this rate of change, that is, the value of the curvature at a given infinitesimally small point.

In the early nineteenth century, when Gauss began to tap into these differential resources, a curved two-dimensional surface was studied using the old Cartesian method: the surface was embedded in a three-dimensional space complete with its own fixed set of axes; then, using those axes, coordinates would be assigned to every point of the surface; finally, the geometric links between points determining the form of the surface would be expressed as algebraic relations between the numbers. But Gauss realized that the calculus, focusing as it does on infinitesimal points on the surface itself (that is, operating entirely with local information), allowed the study of the surface *without any reference to a global embedding space.* Basically, Gauss developed a method to implant the coordinate axes on the surface itself (that is, a method of "coordinatizing" the surface) and, once points had been so translated into numbers, to use differential (not algebraic) equations to characterize their relations. As the mathematician and historian Morris Kline observes, by getting rid of the global embedding space and dealing with the surface through its own local properties "Gauss advanced the totally new concept that *a surface is a space in itself".*[2]

The idea of studying a surface as a space in itself was further developed by Riemann. Gauss had tackled the two-dimensional case, so one would have expected his disciple to treat the next case, three-dimensional curved surfaces. Instead, Riemann went on to successfully attack a much more general problem: that of *N*-dimensional surfaces or spaces. It is these *N*-dimensional curved structures, defined exclusively through their intrinsic features, that were originally referred to by the term "manifold". Riemann's was a very bold move, one that took him into a realm of abstract spaces with a variable number of dimensions, spaces which could be studied without the need to embed them into a higher-dimensional (*N*+1) space. As Morris Kline puts it: "The geometry of space offered by Riemann was not just an extension of Gauss's differential geometry. It reconsidered the whole approach to the study of space."[3] And we could add that this new way of *posing* *spatial problems* would, a few decades later in the hands of Einstein and others, completely alter the way physicists approached the question of space (or more exactly, of spacetime).

A Deleuzian multiplicity takes as its first defining feature these two traits of a manifold: its variable number of dimensions and, more importantly, the absence of a supplementary (higher) dimension imposing an extrinsic coordinatization, and hence, *an extrinsically defined unity.* As Deleuze writes: "Multiplicity must not designate a combination of the many and the one, but rather an organization belonging to the many as such, which has no need whatsoever of unity in order to form a system."[4] Essences, on the other hand, do possess a defining unity (e.g. the unity of rationality and animality defining the human essence) and, moreover, are taken to exist in a transcendent space which serves as a container for them or in which they are embedded. A multiplicity, on the other hand, "however many dimensions it may have, . . . never has a supplementary dimension to that which transpires upon it. This alone makes it natural and immanent."[5] It may be objected that these are purely *formal* differences between concepts, and that as such, they do not necessarily point to a deeper ontological difference. If we are to replace essences as the explanation of the identity of material objects and natural kinds we need to specify the way in which multiplicities relate to the physical processes which generate those material objects and kinds.

Achieving this goal implies establishing a more intimate relation between the geometric properties of manifolds and the properties which define morphogenetic processes. The resources in this case come from the theory of dynamical systems where the dimensions of a manifold are used to represent properties of a particular physical process or system, while the manifold itself becomes *the space of possible states* which the physical system can have.[6] In other words, in this theory manifolds are connected to material reality by their use as *models* of physical processes. When one attempts to model the dynamical behaviour of a particular physical object (say, the dynamical behaviour of a pendulum or a bicycle, to stick to relatively simple cases) the first step is to determine the number of relevant ways in which such an object can *change* (these are known as an object's *degrees of freedom),* and then to relate those changes to one another using the differential calculus. A pendulum, for instance, can change only in its position and momentum, so it has two degrees of freedom. (A pendulum can, of course, be melted at high temperatures, or be exploded by dynamite. These are, indeed, other ways in which this object can change, they simply are not relevant ways from the point of view of dynamics.) A bicycle, if we consider all its moving parts (handlebars, front wheels, crank-chain-rear-wheel assembly and the two pedals) has ten degrees of freedom (each of the five parts can change in both position and momentum).[7]

Next, one maps each degree of freedom into one of the dimensions of a manifold. A pendulum's space of possibilities will need a two-dimensional plane, but the bicycle will involve a ten-dimensional space. After this mapping operation, the state of the object at any given instant of time becomes a *single point* in the manifold, which is now called a *state space.* In addition, we can capture in this model an object's *changes of state* if we allow the representative point to move in this abstract space, one tick of the clock at a time, describing a curve or trajectory. A physicist can then study the changing behaviour of an object by studying the behaviour of these representative trajectories. It is important to notice that even though my example involves two objects, what their state space captures is not their static properties but the way these properties change, that is, *it captures a process.* As with any model, there is a trade-off here: we exchange the complexity of the object's changes of state for the complexity of the modelling space. In other words, an object's instantaneous state, no matter how complex, becomes a single point, a great simplification, but the space in which the object's state is embedded becomes more complex (e.g. the three-dimensional space of the bicycle becomes a ten-dimensional state space).

Besides the great simplification achieved by modelling complex dynamical processes as trajectories in a space of possible states, there is the added advantage that mathematicians can bring new resources to bear to the study and solution of the physical problems involved. In particular, *topological resources* may be used to analyse certain features of these spaces, features which determine *recurrent or typical behaviour* common to many different models, and by extension, common to many physical processes. The main pioneer of this approach was another great nineteenth-century mathematician, Henri Poincaré. Poincaré began his study not with a differential equation modelling a real physical system, but with a very simple equation, so simple it had no physical application, but which nevertheless allowed him to explore the recurrent traits of *any model with two degrees of freedom.* He discovered and classified certain special topological features of two-dimensional manifolds (called *singularities)* which have a large influence in the behaviour of the trajectories, and since the latter represent actual series of states of a physical system, a large influence in the behaviour of the physical system itself.[8]

Singularities may influence behaviour by acting as *attractors* for the trajectories. What this means is that a large number of different trajectories, starting their evolution at very different places in the manifold, may end up in exactly the same final state (the attractor), as long as all of them begin somewhere within the “sphere of influence” of the attractor (the *basin of attraction).* Given that, in this sense, different trajectories may be attracted to the same final state, singularities are said to represent the inherent or intrinsic *long-term tendencies* of a system, the states which the system will spontaneously tend to adopt in the long run as long as it is not constrained by other forces. Some singularities are topological points, so the final state they define as a destiny for the trajectories is a *steady state.* Beside these, Poincaré also found that certain closed loops acted as attractors and called them “limit cycles”. The final state which trajectories attracted to a limit cycle (or periodic attractor) are bound to adopt is an oscillatory state. But whether we are dealing with steady-state, periodic or other attractors what matters is that they are *recurrent topological features,* which means that different sets of equations, representing quite different physical systems, may possess a similar distribution of attractors and hence, similar long-term behaviour.

Let me give a simple example of how singularities (as part of what defines a multiplicity) lead to an entirely different way of viewing the genesis of physical forms. There are a large number of different physical structures which form spontaneously as their components try to meet certain energetic requirements. These components may be constrained, for example, to seek a point of minimal free energy, like a soap bubble, which acquires its spherical form by minimizing surface tension, or a common salt crystal, which adopts the form of a cube by minimizing bonding energy. We can imagine the state space of the process which leads to these forms as structured by a single point attractor (representing a point of minimal energy). One way of describing the situation would be to say that *a topological form* (a singular point in a manifold) guides a process which results in many different physical forms, including spheres and cubes, each one with different *geometric* properties. This is what Deleuze means when he says that singularities are like "implicit forms that are topological rather than geometric".[9] This may be contrasted to the essentialist approach in which the explanation for the spherical form of soap bubbles, for instance, would be framed in terms of the essence of sphericity, that is, of geometrically characterized essences acting as ideal forms.

I will discuss in a moment the meaning and relevance of the topological nature of singularities. What matters at this point is that singularities, by determining long-term tendencies, structure the possibilities which make up state space, and by extension, structure the possibilities open to the physical process modelled by a state space. In addition, singularities tend to be recurrent, that is, they tend to characterize processes independently of their particular physical mechanisms. In the example above, the mechanism which leads to the production of a soap bubble is quite different from the one leading to a salt crystal, yet both are minimizing processes. This *mechanism-independence* is what makes singularities (or rather the multiplicities they define) perfect candidates to replace essences.[10] As I said before, however, we must be careful at this stage not to make singularities the equivalent of the essence of a process. To avoid this error I will discuss some additional formal properties of multiplicities distinguishing them from essences and then, as above, I will discuss the way in which these purely conceptual differences connect with questions of physical process.

The formal difference in question has to do with the way essences and multiplicities are specified as entities. While essences are traditionally regarded as possessing a *clear and distinct* nature (a clarity and distinctiveness also characterizing the ideas which appear in the mind of a philosopher who grasps one of these essences), multiplicities are, by design, *obscure and distinct:* the singularities which define a multiplicity come in sets, and these sets are not given all at once but are structured in such a way that they *progressively specify the nature of a multiplicity* as they unfold following recurrent sequences.[11] What this means may be illustrated first by a metaphor and then given a precise technical definition. The metaphor is that of a fertilized egg prior to it's unfolding into a fully developed organism with differentiated tissues and organs. (A process known as *embryogenesis.)* While in essentialist interpretations of embryogenesis tissues and organs are supposed to be already given in the egg *(preformed,* as it were, and hence having a clear and distinct nature) most biologists today have given up preformism and accepted the idea that differentiated structures emerge progressively as the egg develops. The egg is not, of course, an undifferentiated mass: it possesses an obscure yet distinct structure defined by zones of biochemical concentration and by polarities established by the asymmetrical position of the yolk (or nucleus). But even though it does possess the necessary biochemical materials and genetic information, these materials and information do not contain a clear and distinct blueprint of the final organism.[12]

Although the egg metaphor does provide a vivid illustration of the distinction I am trying to draw here, it is nevertheless just a useful analogy. Fortunately, there are technical ways of defining the idea of *progressive differentiation* which do not rely on metaphors. The technical resources in this case come from another crucial nineteenth-century innovation, the theory of groups, a field of mathematics which, like the differential geometry I discussed before, eventually became an integral part of the basic mathematical technology of twentieth-century physics. The term “group” refers to a set of entities (with special properties) and a rule of combination for those entities. The most important of the properties is the one named “closure”, which means that, when we use the rule to combine any two entities in the set, the result is an entity also belonging to the set. For example, the set of positive integers displays closure if we use addition as a combination rule: adding together any two positive integers yields another positive integer, that is, another element in the original set.[13]

Although sets of numbers (or many other mathematical objects) may be used as illustrations of groups, for the purpose of defining progressive differentiation we need to consider groups whose members are not objects but *transformations* (and the combination rule, a consecutive application of those transformations). For example, the set consisting of rotations by ninety degrees (that is, a set containing rotations by 0, 90, 180, 270 degrees) forms a group, since any two consecutive rotations produce a rotation also in the group, provided 360 degrees is taken as zero. The importance of groups of transformations is that they can be used to classify geometric figures by their *invariants:* if we performed one of this group's rotations on a cube, an observer who did not witness the transformation would not be able to notice that any change had actually occurred (that is, the visual appearance of the cube would remain invariant relative to this observer). On the other hand, the cube would not remain invariant under rotations by, say, 45 degrees, but a sphere would. Indeed, a sphere remains visually unchanged under rotations by *any amount* of degrees. Mathematically this is expressed by saying that the sphere has *more symmetry* than the cube relative to the rotation transformation. That is, degree of symmetry is measured by the number of transformations in a group that leave a property invariant, and relations between figures may be established if the group of one is included in (or is a subgroup of) the group of the other.

Classifying geometrical objects by their degrees of symmetry represents a sharp departure from the traditional classification of geometrical figures by their essences. While in the latter approach we look for a set of properties common to all cubes, or to all spheres, groups do not classify these figures on the basis of their static properties but in terms of how these figures are affected (or not affected) by active transformations, that is, figures are classified by *their response to events that occur to them*.[14] Another way of putting this is that even though in this new approach we are still classifying entities by a property (their degree of symmetry), this property is never an intrinsic property of the entity being classified but always a property relative to a specific transformation (or group of transformations). Additionally, the symmetry approach allows dynamic relations to enter into the classification in a different way. When two or more entities are related as the cube and the sphere above, that is, when the group of transformations of one is a subgroup of the other, it becomes possible to envision *a process which converts one of the entities into the other* by losing or gaining symmetry. For example, a sphere can "become a cube" by loosing invariance to some transformations, or to use the technical term, by undergoing a *symmetry-breaking transition.* While in the realm of pure geometry this transmutation may seem somewhat abstract, and irrelevant to what goes on in the worlds of physics or biology, there are many illustrations of symmetry-breaking transitions in these more concrete domains.

In a physical process transmutations through broken symmetry may occur, for example, in the form of *phase transitions.* Phase transitions are events which take place at critical values of some parameter (temperature, for example) switching a physical system from one state to another, like the critical points of temperature at which water changes from ice to liquid, or from liquid to steam. The broken symmetry aspect here can be clearly seen if we compare the gas and solid states of a material, and if for simplicity we assume perfectly uniform gases and perfect crystal arrangements. In these ideal conditions, the gas would display invariant properties under all translations, rotations and reflections, while the solid would be invariant to only a subset of these transformations. For example, while the gas could be displaced by any amount and remain basically the same (that is, an observer would be unable to tell whether a displacement occurred at all) the solid would remain visually unchanged only under displacements which moved it one unit crystal at a time (or multiples of that unit). In other words, the gas has more symmetry than the solid, and can become the solid by undergoing a symmetry-breaking phase transition.[15] The metaphorical example I gave above, that of a fertilized egg which differentiates into a fully formed organism, can now be made quite literal: the progressive differentiation of the spherical egg is achieved through a complex cascade of symmetry-breaking phase transitions.[16]

Let me now incorporate the idea of progressive differentiation into the concept of multiplicity by showing how it can be translated into state-space terms. I said before that for the purpose of defining an entity to replace essences the aspect of state space that mattered was its singularities. One singularity (or set of singularities) may undergo a symmetry-breaking transition and be converted into another one. These transitions are called *bifurcations* and may be studied by adding to a particular state space one or more “control knobs” (technically, control parameters) which determine the strength of external shocks or perturbations to which the system being modelled may be subject. These control parameters tend to display *critical values,* thresholds of intensity at which a particular bifurcation takes place breaking the prior symmetry of the system. A state space structured by one point attractor, for example, may bifurcate into another with two such attractors, or a point attractor may bifurcate into a periodic one, losing some of its original symmetry.[17] Much as attractors come in recurrent forms, so bifurcations may define *recurrent sequences* of such forms. There is a sequence, for instance, that begins with a point attractor which, at a critical value of a control parameter, becomes unstable and bifurcates into a periodic attractor. This cyclic singularity, in turn, can become unstable at another critical value and undergo a sequence of instabilities (several period-doubling bifurcations) which transform it into a chaotic attractor.

This symmetry-breaking cascade of bifurcations can, in turn, be related to actual recurring sequences in physical processes. There is, for example, a realization of the above cascade occurring in a well-studied series of distinct hydrodynamic flow patterns (steady-state, cyclic and turbulent flow). Each of these recurrent flow patterns appears one after the other at well-defined critical thresholds of temperature or speed. The sequence of phase transitions may be initiated by heating a water container from below. At low temperatures the flow of heat from top to bottom, referred to as thermal *conduction,* is simple and steady, displaying only a bland, featureless overall pattern, having the degree of symmetry of a gas. At a critical point of temperature, however, this steady flow suddenly disappears and another one takes its place, thermal *convection,* in which coherent rolls of water form, rotating either clockwise or anti-clockwise. The water container now has structure and, for the same reason, has lost some symmetry. As the temperature continues to intensify another threshold is reached, the flow loses its orderly periodic form and a new pattern takes over: *turbulence.* The cascade that yields the sequence conduction—convection—turbulence is, indeed, more complicated and may be studied in detail through the use of a special machine called the Couette–Taylor apparatus, which speeds up (rather than heats up) the liquid material. At least seven different flow patterns are revealed by this machine, each appearing at a specific critical point in speed, and thanks to the simple cylindrical shape of the apparatus, each phase transition may be directly related to a broken symmetry in the group of transformations of the cylinder.[18]

As can be seen from this example, a cascade of bifurcations may be faithfully realized in a physical system. This realization, however, bears no resemblance to the mathematical cascade. In particular, unlike the latter which is *mechanism-independent,* the physical realization involves specific mechanisms. To begin with there are causal interactions and their effects. To return to our example, the flow of heat into the container causes a graded density difference to form, given that water expands when heated (that is, becomes less dense). This density gradient, in turn, interacts with other forces like the viscosity of the water, their balance of power determining whether a system switches from one flow pattern to the next. For example, the density gradient will tend to amplify small differences in movement (fluctuations) which could add some detail to the bland steady-state flow, but which are damped by the viscosity of the fluid. As the flow of heat is intensified, however, the system reaches a critical point at which the density gradient is strong enough to overcome viscosity, leading to the amplification of fluctuations and allowing the formation of coherent rolls. Thus, a very specific sequence of events underlies the transition to convection. On the other hand, as the biologist Brian Goodwin has pointed out, portions of this hydrodynamic sequence may be observed in a completely different process, the complex morphogenetic sequence which turns a fertilized egg into a fully developed organism. After describing another instance of a sequence of flow patterns in hydrodynamics Goodwin says:

> The point of the description is not to suggest that morphogenetic patterns originate from the hydrodynamic properties of living organisms . . . What I want to emphasize is simply that many pattern-generating processes share with developing organisms the characteristic that spatial detail unfolds progressively simply as a result of the laws of the process. In the hydrodynamic example we see how an initially smooth fluid flow past a barrier goes through a symmetry-breaking event to give a spatially periodic pattern, followed by the elaboration of local nonlinear detail which develops out of the periodicity. Embryonic development follows a similar qualitative course: initially smooth primary axes, themselves the result of spatial bifurcation from a uniform state, bifurcate to spatially periodic patterns such as segments [in an insect body], within which finer detail develops . . . through a progressive expression of nonlinearities and successive bifurcations . . . The role of gene products in such an unfolding is to stabilize a particular morphogenetic pathway by facilitating a sequence of pattern transitions, resulting in a particular morphology.[19]

From a Deleuzian point of view, it is this *universality* (or mechanism-independence) of multiplicities which is highly significant. Unlike essences which are always abstract and general entities, multiplicities are *concrete universals.* That is, concrete sets of attractors (realized as tendencies in physical processes) linked together by bifurcations (realized as abrupt transitions in the tendencies of physical processes). Unlike the generality of essences, and the resemblance with which this generality endows instantiations of an essence, the universality of a multiplicity is typically *divergent:* the different realizations of a multiplicity bear no resemblance whatsoever to it and there is in principle no end to the set of potential divergent forms it may adopt. This lack of resemblance is amplified by the fact that multiplicities give form to processes, not to the final product, so that the end results of processes realizing the same multiplicity may be highly dissimilar from each other, like the spherical soap bubble and the cubic salt crystal which not only do not resemble one another, but bear no similarity to the topological point guiding their production.

The concept of progressive differentiation which I have just defined was meant, as I said, to distinguish the obscure yet distinct nature of multiplicities from the clear and distinct identity of essences, as well as from the clarity afforded by the light of reason to essences grasped by the mind. A final distinction must now be made: unlike essences, which as abstract general entities coexist side by side sharply distinguished from one another, concrete universals must be thought as *meshed together into a continuum.* This further blurs the identity of multiplicities, creating zones of indiscernibility where they blend into each other, forming a continuous immanent space very different from a reservoir of eternal archetypes. Multiplicities, as Deleuze writes, coexist

> but they do so at points, on the edges, and under glimmerings which never have the uniformity of a natural light. On each occasion, obscurities and zones of shadow correspond to their distinction. [Multiplicities] are distinguished from one another, but not at all in the same manner as forms and the terms in which these are incarnated. They are objectively made and unmade according to the conditions that determine their fluent synthesis. This is because they combine the greatest power of being differentiated with an inability to be differenciated.[20]

Although I will not stick to this subtle typographical distinction, Deleuze distinguishes the progressive unfolding of a multiplicity through broken symmetries (differentiation), from the progressive specification of the continuous space formed by multiplicities as it gives rise to our world of discontinuous spatial structures (differenciation). Unlike a transcendent heaven which exists as a *separate dimension* from reality, Deleuze asks us to imagine a continuum of multiplicities which *differenciate itself* into our familiar three-dimensional space as well as its spatially structured contents.

Let me explain in what sense a continuous space may be said to become progressively defined giving rise to discontinuous spaces. First of all, a space is not just a set of points, but a set together with a way of binding these points together into *neighbourhoods* through well-defined relations of *proximity or contiguity.* In our familiar Euclidean geometry these relations are specified by fixed lengths or distances which determine how close points are to each other. The concept of "length" (as well as related ones, like "area" or "volume") is what is called a *metric* concept, so the spaces of Euclidean geometry are known as *metric spaces.*[21] There exist other spaces, however, where fixed distances cannot define proximities since distances do not remain fixed. A topological space, for example, may be stretched without the neighbourhoods which define it changing in nature. To cope with such exotic spaces, mathematicians have devised ways of defining the property of "being nearby" in a way that does not presuppose any metric concept, but only nonmetric concepts like "infinitesimal closeness". However one characterizes it, the distinction between *metric and nonmetric spaces* is fundamental in a Deleuzian ontology.[22] Moreover, and this is the crucial point, there are well-defined technical ways of linking metric and non-metric spaces in such a way that the former become the product of the progressive differentiation of the latter. To explain how such a symmetry-breaking cascade would work in this case, I will need to take a brief detour through the history of nineteenth-century geometry.

Although in that century most physicists and mathematicians thought the structure of physical space was captured by Euclidean geometry, many other geometries, with very different properties, had come into existence. Some of them (such as the non-Euclidean geometry developed by Lobatchevsky) shared with the geometry of Euclid the property of being metric. There were, however, other geometries where metric concepts were not in fact fundamental. The differential geometry of Gauss and Riemann which gave us the concept of a manifold is one example, but there were several others (projective geometry, affine geometry, topology). Moreover, and despite the fact that Euclidean geometry reigned supreme, some mathematicians realized that its basic concepts could in fact be *derived* from the nonmetric concepts which formed the foundation of the newcomers. In particular, another influential nineteenth-century mathematician, Felix Klein, realized that all the geometries known to him could be categorized by their invariants under groups of transformations, and that the different groups were embedded one into the other.[23] In modern terminology this is equivalent to saying that the different geometries were related to each other by relations of broken symmetry.

In Euclidean geometry, for example, lengths, angles and shapes remain unaltered by a group containing rotations, translations and reflections. This is called the group of *rigid transformations.* These metric properties, however, do not remain invariant under the groups of transformations characterizing other geometries. There is one geometry, called *affine geometry,* which adds to the group characterizing Euclidean geometry new transformations, called *linear transformations,* under which properties like the parallelism or the straightness of lines remain invariant, but not their lengths. Then there is *projective geometry,* which adds to rigid and linear transformations those of projection, corresponding to shining light on a piece of film, and section, the equivalent of intercepting those light rays on a screen. (More technically, this geometry adds transformations called “projectivities”.) These transformations do not necessarily leave Euclidean or affine properties unchanged, as can be easily pictured if we imagine a film projector (which typically increases the magnitude of lengths) and a projection screen at an angle to it (which distorts parallel lines).

If we picture these three geometries as forming the levels of a hierarchy (projective—affine—Euclidean) it is easy to see that the transformation group of each level includes the transformations of the level below it and adds new ones. In other words, each level possesses more symmetry than the level below it. This suggests that, as we move down the hierarchy, a symmetry-breaking cascade should produce progressively more differentiated geometric spaces, and, vice versa, that as we move up we should lose differentiation. For example, as we ascend from Euclidean geometry more and more figures become equivalent to one another, forming *a lesser number of distinct classes.* Thus, while in Euclidean geometry two triangles are equivalent only if their sides have the same length, in affine geometry all triangles are the same (regardless of lengths). In other words, as we move up the class of equivalent triangles becomes less differentiated. Or to take a different example, while in Euclidean geometry two conic sections (the family of curves containing circles, ellipses, parabolas and hyperbolas) are equivalent if they are both of the same type (both circles or both parabolas) and have the same size, in affine geometry they only need to be of the same type (regardless of size) to be equivalent, while in projective geometry all conic sections, without further qualification, are the same.[24] In short, as we move up the hierarchy figures which used to be fully differentiated from one another become progressively less distinct eventually blending into a single one, and vice versa, as we move down, what used to be one and the same shape progressively differentiates into a variety of shapes.

This hierarchy can be expanded to include other geometries, such as differential geometry and topology. The latter, for example, may be roughly said to concern the properties of geometric figures which remain invariant under bending, stretching, or deforming transformations, that is, transformations which do not create new points or fuse existing ones. (More exactly, topology involves transformations, called "homeomorphisms", which *convert nearby points into nearby points* and which can be reversed or be continuously undone.) Under these transformations many figures which are completely distinct in Euclidean geometry (a triangle, a square and a circle, for example) become one and the same figure, since they can be deformed into one another. In this sense, topology may be said to be the *least differentiated* geometry, the one with the least number of distinct equivalence classes, the one in which many discontinuous forms have blended into one continuous one.[25] Metaphorically, the hierarchy "topological—differential—projective—affine—Euclidean" may be seen as representing an abstract scenario for the birth of real space. As if the metric space which we inhabit and that physicists study and measure *was born* from a nonmetric, topological continuum as the latter differentiated and acquired structure following a series of symmetry-breaking transitions.

This *morphogenetic* view of the relation between the different geometries is a metaphor in the sense that to mathematicians these relations are purely *logical,* useful because theorems which are valid at one level are automatically valid at the levels below it.[26] But this cascade of broken symmetries may be also given *an ontological dimension.* One way in which this scenario for the birth of metric space can be made less metaphorical and more directly ontological, is through a comparison between metric and nonmetric geometrical properties, on one hand, and *extensive and intensive physical properties,* on the other. Extensive properties include not only such metric properties as length, area and volume, but also quantities such as amount of energy or entropy. They are defined as properties which are *intrinsically divisible:* if we divide a volume of matter into two equal halves we end up with two volumes, each half the extent of the original one. Intensive properties, on the other hand, are properties such as temperature or pressure, which cannot be so divided. If we take a volume of water at 90 degrees of temperature, for instance, and break it up into two equal parts, we do not end up with two volumes at 45 degrees each, but with two volumes at the original temperature.[27]

Deleuze argues, however, that an intensive property is not so much one that is indivisible but one which *cannot be divided without involving a change in kind.*[28] The temperature of a given volume of liquid water, for example, can indeed be "divided" by heating the container from underneath creating a temperature difference between the top and bottom portions of the water. Yet, while prior to the heating the system is at equilibrium, once the temperature difference is created the system will be away from equilibrium, that is, we can divide its temperature but in so doing we change the system qualitatively. Indeed, as we just saw, if the temperature difference is made intense enough the system will undergo a phase transition, losing symmetry and changing its dynamics, developing the periodic pattern of fluid motion which I referred to above as "convection". Thus, in a very real sense, phase transitions do divide the temperature scale but in so doing they mark sudden changes in the spatial symmetry of a material.

Using these new concepts we can define the sense in which the metric space we inhabit emerges from a nonmetric continuum through a cascade of broken symmetries. The idea would be to view this genesis not as an abstract mathematical process but as a concrete physical process in which an undifferentiated *intensive space* (that is, a space defined by continuous intensive properties) progressively differentiates, eventually giving rise to *extensive structures* (discontinuous structures with definite metric properties). We can take as an illustration of this point some recent developments in quantum field theories. Although the concept of spontaneous symmetry breaking, and its connection with phase transitions, developed in rather humble branches of physics, like the fields of hydrodynamics and condensed matter physics, it was eventually incorporated into the main stream.[29] Today, this concept is helping unify the four basic forces of physics (gravitational, electromagnetic, strong and weak nuclear forces) as physicists realize that, at extremely high temperatures (the extreme conditions probably prevailing at the birth of the universe), these forces lose their individuality and blend into one, highly symmetric, force. The hypothesis is that as the universe expanded and cooled, a series of phase transitions broke the original symmetry and allowed the four forces to differentiate from one another.[30] If we consider that, in relativity theory, gravity is what gives space its metric properties (more exactly, a gravitational field constitutes the metric structure of a four-dimensional manifold), and if we add to this that gravity itself emerges as a distinct force at a specific critical point of an intensive property (temperature), the idea of an intensive space giving birth to extensive ones through progressive differentiation becomes more than a suggestive metaphor.[31]

Let me pause for a moment to summarize the argument so far. I began by establishing some purely formal differences between the concepts of "essence" and of "multiplicity": while the former concept implies a unified and timeless identity, the latter lacks unity and implies an identity which is not given all at once but is defined progressively; and while essences bear to their instantiations the same relation which a model has to its copies, that is, a relation of greater or lesser resemblance, multiplicities imply divergent realizations which bear no similarity to them. These formal differences, I said, are insufficient to characterize the distinction between essences and multiplicities as *immaterial entities* whose job is to account for the genesis of form: replacing eternal archetypes involves supplying an alternative explanation of morphogenesis in the world. Unlike essences which assume that matter is a passive receptacle for external forms, multiplicities are immanent to material processes, defining their spontaneous capacity to generate pattern without external intervention. I used certain features of mathematical models (state spaces) to define the nature of multiplicities: a multiplicity is defined by distributions of singularities, defining tendencies in a process; and by a series of critical transitions which can take several such distributions embedded within one another and unfold them. Finally, I said that a population of such concrete universals forms a real dimension of the world, a nonmetric continuous space which progressively specifies itself giving rise to our familiar metric space as well as the discontinuous spatial structures that inhabit it.

No doubt, despite my efforts these remarks remain highly metaphorical. First of all, I have defined multiplicities in terms of attractors and bifurcations but these are features of mathematical models. Given that I want the term "multiplicity" to refer to a concrete universal (to replace abstract general essences) the question may arise as to the legitimacy of taking features of a model and reifying them into the defining traits of a real entity. Second, the relation between a continuum of multiplicities and the discontinuous and divisible space of our everyday world was specified entirely by analogy with a purely mathematical construction, the hierarchy of geometries first dreamt by Felix Klein. Eliminating the metaphorical content will involve not only a thorough ontological analysis of state space so that its *topological invariants* can be separated from its variable mathematical content, but in addition, a detailed discussion of how these topological invariants may be woven together *to construct* a continuous, yet heterogeneous, space. In the following chapter I will show in technical detail how this construction can be carried out and how the resulting continuum may replace the top or least metric level in the hierarchy of geometries. I will also discuss how the intermediate levels may be replaced by intensive processes of individuation which yield as their final product the fully differentiated metric structures that populate the bottom level. At the end of chapter two the metaphor of a genesis of metric space through a cascade of broken symmetries should have been mostly eliminated, and a literal account taken its place.

Meanwhile, in what remains of this chapter I would like to make a more detailed analysis of the nature of multiplicities. The first set of issues to be discussed will involve the technical details of Deleuze's ontological interpretation of the contents of state space. His approach is very unorthodox as will be shown by a comparison with the state space ontologies proposed by analytical philosophers. Then I will move on to a second set of issues concerning the *modal status* of multiplicities. Modal logic is the branch of philosophy which deals with the relations between *the possible and the actual.* Here the question to be answered is if state space is a space of possible states what is the status of attractors and bifurcations in relation to these possibilities? Can multiplicities be interpreted in terms of the traditional modal categories, the possible and the necessary, or do we need to postulate an original form of physical modality to characterize them? Finally, a third set of issues that needs to be dealt with is related to the speculative dimension of Deleuze's project. Replacing essences with social conventions or subjective beliefs is a relatively safe move, but putting in their place a new set of objective entities inevitably involves philosophical speculation. What guides this speculation? One way of looking at this question is to see Deleuze as engaged in a constructive project guided by certain *proscriptive constraints,* that is, constraints which tell him not what to do but what to avoid doing. One such constraint is, of course, to avoid the trap of essentialism, but there are others and these need to be discussed.

Let me begin with Deleuze's ontological analysis of state space. Many philosophers are today looking at these abstract spaces as objects of study and reflection. A recent shift in the analytical philosophy of science, for example, moving away from logic (and set theory) and towards an analysis of the actual mathematics used by scientists in their everyday practice, has brought the importance of state spaces to the foreground.[32] Yet none of the philosophers involved in this new movement has attempted such an original analysis of state space as Deleuze has. In particular, analytical philosophers seem unaware of (or at least unconcerned with) Poincaré's topological studies and of the ontological difference that may be posited between the recurrent features of state space and the trajectories these features determine. Given that this ontological difference is key to the idea of a Deleuzian multiplicity, I will need to explain how state spaces are constructed. First of all, it is important to distinguish the different operators involved in this construction. As I said above, given a relation between the changes in two (or more) degrees of freedom expressed as a rate of change, one operator, differentiation, gives us the instantaneous value for such a rate, such as an instantaneous velocity (also known as a *velocity vector).* The other operator, integration, performs the opposite but complementary task: from the instantaneous values it reconstructs a full trajectory or series of states.

These two operators are used in a particular order to generate the structure of state space. The modelling process begins with a choice of manifold to use as a state space. Then from experimental observations of a system's changes in time, that is, from actual series of states as observed in the laboratory, we create some trajectories to begin populating this manifold. These trajectories, in turn, serve as the raw material for the next step: we repeatedly apply the differentiation operator to the trajectories, each application generating one velocity vector and in this way we generate a *velocity vector field.* Finally, using the integration operator, we generate from the vector field further trajectories which can function as predictions about future observations of the system's states. The state space filled with trajectories is called the "phase portrait" of the state space.[33] Deleuze makes a *sharp ontological distinction between the trajectories* as they appear in the phase portrait of a system, on one hand, *and the vector field,* on the other. While a particular trajectory (or integral curve) models a succession of actual states of a system in the physical world, the vector field captures the inherent tendencies of many such trajectories, and hence of many actual systems, to behave in certain ways. As mentioned above, these tendencies are represented by singularities in the vector field, and as Deleuze notes, despite the fact that the *precise nature* of each singular point is well defined only in the phase portrait (by the form the trajectories take in its vicinity) *the existence and distribution* of these singularities is already completely given in the vector (or direction) field. In one mathematician's words:

> The geometrical interpretation of the theory of differential equations clearly places in evidence two absolutely distinct realities: there is the field of directions and the topological accidents which may suddenly crop up in it, as for example the existence of . . . singular points to which no direction has been attached; and there are the integral curves with the form they take on in the vicinity of the singularities of the field of directions . . . The existence and distribution of singularities are notions relative to the field of vectors defined by the differential equation. The form of the integral curves is relative to the solution of this equation. The two problems are assuredly complementary, since the nature of the singularities of the field is defined by the form of the curves in their vicinity. But it is no less true that the field of vectors

on one hand and the integral curves on the other are two essentially distinct mathematical realities.[34]

There are several other features of singularities, or more specifically, of attractors, which are crucial in an ontological analysis of state space, and which further differentiate its two "distinct mathematical realities". As is well known, the trajectories in this space always approach an attractor *asymptotically,* that is, they approach it *indefinitely close but never reach it.*[35] This means that unlike trajectories, which represent the actual states of objects in the world, attractors are *never actualized,* since no point of a trajectory ever reaches the attractor itself. It is in this sense that singularities represent only the long-term tendencies of a system, never its actual states. Despite their lack of actuality, attractors are nevertheless real and have definite effects on actual entities. In particular, they confer on trajectories a certain degree of stability, called *asymptotic stability.*[36] Small shocks may dislodge a trajectory from its attractor but as long as the shock is not too large to push it out of the basin of attraction, the trajectory will naturally return to the stable state defined by the attractor (a steady state in the case of point attractors, a stable cycle in the case of periodic attractors, and so on). Another important feature involves not the stability of the trajectories but that of the distribution of attractors itself (its *structural stability*). Much as the stability of trajectories is measured by their resistance to small shocks, so the stability of a particular distribution of attractors is checked by submitting the vector field to perturbations, an effect achieved by adding a small vector field to the main one, and checking whether the resulting distribution of attractors is *topologically equivalent* to the original one.[37] Typically, distributions of attractors are structurally stable and this, in part, is what accounts for their recurrence among different physical systems. On the other hand, if the perturbation is large enough a distribution of attractors may cease to be structurally stable and change or bifurcate into a different one. Such a bifurcation event is defined as a continuous deformation of one vector field into another topologically *inequivalent* one through a structural *instability*.[38]

Using the technical terms just introduced I can give now a final definition of a multiplicity. A multiplicity *is a nested set of vector fields related to each other by symmetry-breaking bifurcations, together with the distributions of attractors which define each of its embedded* *levels.* This definition separates out the part of the model which carries information about the actual world (trajectories as series of possible states) from that part which is, in principle, *never actualized.* This definition presupposes only the two concepts of "differential relation" and "singularity". I will return in the next chapter to a discussion of what further *philosophical transformation* these two concepts need to undergo in order to be truly detached from their mathematical realization. At this point, granting that the definition I just gave could specify a concrete entity, we may ask what ontological status such an entity would have? To speak as I did of patterns of hydrodynamic flow and of patterns of embryological development as divergent *realizations* of a universal multiplicity is misleading since it suggests that these patterns are real, while the multiplicity itself is not. So Deleuze speaks not of "realization" but of *actualization,* and introduces a novel ontological category to refer to the status of multiplicities themselves: *virtuality.* This term does not refer, of course, to the virtual reality which digital simulations have made so familiar, but to a *real virtuality* forming a vital component of the objective world. As he writes:

> The virtual is not opposed to the real but to the actual. The virtual is fully real in so far as it is virtual . . . Indeed, the virtual must be defined as strictly a part of the real object—as though the object had one part of itself in the virtual into which it plunged as though into an objective dimension . . . The reality of the virtual consists of the differential elements and relations along with the singular points which correspond to them. The reality of the virtual is structure. We must avoid giving the elements and relations that form a structure an actuality which they do not have, and withdrawing from them a reality which they have.[39]

What is *the modal status* of the virtual? If state space trajectories have the status of possibilities (possible series of states) what modality do virtual multiplicities represent? This is not an easy question to answer given that the ontological status of even the familiar modal categories is a thorny issue. So before dealing with virtuality let me discuss the question of possibility. Traditionally, ontological discussion of possibilities has been very controversial due to their elusive nature, and in particular, to the difficulty of giving a clear criterion for *individuating* them, that is, for telling when we have one instead of another possibility. As a famous critic of modal logic, the philosopher Willard Van Orman Quine, jokes:

> Take, for instance, the possible fat man in the doorway; and again, the possible bald man in the doorway. Are they the same possible man, or two possible men? How do we decide? How many possible men there are in that doorway? Are there more possible thin ones than fat ones? How many of them are alike? Or would their being alike make them one? Are not two possible things alike? Is this the same as saying that it is impossible for two things to be alike? Or, finally, is the concept of identity simply inapplicable to unactualized possibles? But what sense can be found in talking of entities which cannot be meaningfully said to be identical with themselves and distinct from one another?[40]

Most approaches to modal logic concentrate on language, or more specifically, on an analysis of sentences which express *what could have been,* sentences such as "If J.F.K. had not been assassinated then the Vietnam War would have ended sooner." Given that human beings seem capable of routinely using and making sense of these counterfactual sentences, the modal logician's task is to explain this ordinary capability.[41] However, the fact that linguistically specified *possible worlds* (like the possible world where J.F.K. survived) are so devoid of structure, and allow so much ambiguity as to what distinguishes one possible world from another, is what has prompted criticisms such as Quine's. But as some philosophers have suggested, the problem here would seem to be with linguistic representations and their lack of resources to structure possible worlds, and not with possibilities as such. The philosopher of science Ronald Giere, for instance, thinks the extra constraints which structure state space can overcome the limitations of other modal approaches:

> As Quine delights in pointing out, it is often difficult to individuate possibilities . . . [But] many models in which the system laws are expressed as differential equations provide an unambiguous criterion to individuate the possible histories of the model. They are the trajectories in state space corresponding to all possible initial conditions. Threatened ambiguities in the set of possible initial

conditions can be eliminated by explicitly restricting the set in the definition of the theoretical model.[42]

Giere argues that state spaces may be viewed as a way of specifying possible worlds for a given physical system, or at least, possible histories for it, each trajectory in the phase portrait representing one possible historical sequence of states for a system or process. The individuality of the different possible histories within state space is defined by *laws,* expressed by the differential equations that functionally relate the system's degrees of freedom, as well as by *initial conditions,* the specific state, or point in the manifold, where a system begins its evolution. Given a specific initial condition and a deterministic law (such as those of classical physics) one and only one trajectory is individuated, a fact that may be used to challenge Quine's sceptical stance. The phase portrait of any particular state space will be typically filled with many such individual trajectories, one for each possible initial condition. One may reduce this number by adding other laws which forbid certain combinations of values for the degrees of freedom, that is, which make some initial conditions not available for a given system, but still, one ends up with many possible histories.[43]

The problem for the philosopher becomes what *ontological status* to assign to these well-defined possibilities. One ontological stance, which Giere calls "actualism", denies any reality to the possible trajectories, however well individuated they may be. A mathematical model, in this view, is simply a tool to help us in the control of particular physical systems (that is, the manipulation in the laboratory of the behaviour of real systems) as well as in the prediction of their future behaviour. For this limited purpose of prediction and control all we need to judge is the *empirical adequacy* of the model: we generate *one trajectory* for a given initial condition, then try to reproduce that particular combination of values for the degrees of freedom in the laboratory, and observe whether the sequence of *actual states* matches that predicted by the trajectory. Given the one trajectory we associate with the actual sequence in an experiment, the rest of the population of trajectories is merely a useful fiction, that is, ontologically unimportant.[44] As Giere argues, however, this ontological stance misses the fact that the population of trajectories as a whole *displays certain regularities* in the possible histories of a system, global regularities which play a role in shaping any one particular actual history.[45] To him, understanding a system is not knowing how it actually behaves in this or that specific situation, but knowing *how it would behave* in conditions which may in fact not occur. And to know that we need to use the global information embodied in the population of possible histories, information which is lost if we concentrate on the one trajectory which is compared with real sequences of states.[46]

As should be clear from the discussion in this chapter, Deleuze was not an "actualist". He held a realist position towards the modal structure of state space but would have disagreed with Giere in his interpretation of what constitutes that modal structure. In particular, in a Deleuzian ontology one must emphasize that the regularities displayed by the different possible trajectories are *a consequence* of the singularities that shape the vector field. The well-defined nature of the possible histories is not to be approached by a mere mention of laws expressed as differential equations, but by an understanding of how such equations in fact individuate trajectories. Each possible sequence of states, each possible history, is generated by following at each point of the trajectory the directions specified by the vector field, and any regularities or propensities exhibited by the trajectories should indeed be ascribed to the topological accidents or singularities of the field of directions. As Deleuze puts it, "the singularities preside over the genesis" of the trajectories.[47] In other words, Giere is right in thinking that state space offers more resources than language to individuate possibilities (thus sidestepping Quine's criticisms) but wrong in his assessment of how the *process of individuation* takes place. To leave the vector field out of our ontological analysis (that is, to make it into an auxiliary construction or yet another useful fiction) hides the real source of the regularities or propensities in the population of possible histories.[48]

This point tends to be obscured in traditional philosophical analyses by the use of examples involving the simplest type of equation, *a linear equation.* Despite the fact that of all the types of equations available to physicists the linear type is *the least typical,* it happens to be the type that became dominant in classical physics. The vector fields of these differential equations are extremely simple, "the only possible attractor of a linear dynamical system is a fixed point. Furthermore, this fixed point is unique—a linear dynamical system cannot have more than one basin of attraction."[49] In other cases (in conservative systems which are quasi-isolated from their surroundings) there may be no attractors at all, only trajectories. Thus, in a linear conservative system (such as the harmonic oscillator used as an example by Giere) the vector field is so barely structured that it may, for most practical purposes, be ignored as a source of constraints in the individuation of trajectories. On the other hand, the more typical equations (nonlinear equations) have a more elaborate distribution of singularities, the state space being normally partitioned in a cellular fashion by many attractors and their basins, and these multiple attractors may be of different types. In these more common cases, the vector field has too much structure to be ignored.[50]

This argument, however, establishes only that there are in state space other constraints for the individuation of possible histories, but not that they should be given a separate modal status. We could, it would seem, take singularities to belong to the realm of the possible and save ourselves the trouble of introducing novel forms of physical modality, such as virtuality. One way of doing this would be to take a basin of attraction to be merely a subset of points of state space. Given that state space is a space of possible states, any subset of it will also be just a collection of possibilities.[51] Yet, as I mentioned before, despite the fact that the nature of singularities is well defined only in the phase portrait of a system, their *existence and distribution* is already given in the vector field, where they define overall flow tendencies for the vectors. It may seem plausible to think of point attractors, for example, as just one more point of state space, but this singular point is not an available possibility for the system since it is never occupied by a trajectory, only approached by it asymptotically. Trajectories will tend to approach it ever closer but never reach it, and even when one speaks of the end state of a trajectory, in reality the curve is fluctuating around its attractor, not occupying it. Strictly speaking, as I said above, attractors are *never actualized.*

Thus, it seems, a more complete analysis of state space does seem to demand a form of physical modality that goes beyond mere possibility. But could not that other traditional modal category, *necessity,* do the job? After all, in classical physics' models a general law relates all the successive points of a trajectory in a necessary or deterministic way, and which specific trajectory is generated is necessarily determined given a particular initial state.[52] This is, indeed, true, but the relative importance of general laws and particular initial conditions changes once we add singularities. On one hand, the role of any particular initial state is greatly diminished since many initial conditions (all those that are included within a particular basin) will be equivalent as far as the end state of the trajectory is concerned. The states a trajectory adopts on its way to the end state, what engineers call its transient states and which constitute the bulk of the trajectory, may be of interest sometimes, but clearly will not be as important as the stable end state, since the system will spend most of its time fluctuating around that state. On the other hand, the role of the general law will also be diminished because the behaviour of the trajectory at its end state, a steady-state or a cyclic behaviour, for example, will be determined not by its previous states (defined by the general law), but by the type of the attractor itself.

This argument, again, establishes the need to consider additional factors in the individuation of possible histories but not the need for additional modalities. After all, is not the end state of a trajectory necessary? In this case too, the complexity of the distribution of singularities makes a great difference in our interpretation of the modal structure of state space. A state space with a single attractor, and a single basin encompassing the entire space, has a unique end state for the evolution of the system. Concentrating on this atypical case, therefore, can mislead us into thinking that determinism implies a single necessary outcome. On the other hand, a space with multiple attractors *breaks the link between necessity and determinism,* giving a system a “choice” between different destinies, and making the particular end state a system occupies a combination of determinism and chance. For instance, which attractor a system happens to be in at any one time is determined, in part, by its contingent history: a trajectory may be dislodged from an attractor by an *accident,* a strong-enough external shock pushing it out of one basin and into the sphere of influence of another attractor. Furthermore, which specific distribution of attractors a system has available at any one point in its history, may be changed by a bifurcation. When a bifurcation leads to two alternative distributions, only one of which can be realized, a deterministic system faces further “choices”. Which alternative obtains, as nonlinear scientists Ilya Prigogine and Gregoire Nicolis have been arguing for decades, will be decided by chance fluctuations in the environment. Speaking of the emergence of convection cells at a phase transition, these authors write:

> As soon as [the critical value is reached] we know that the cells will appear: this phenomenon is therefore subject to strict determinism.

> In contrast, the direction of rotation of the cells [clock- or anti-clockwise] is unpredictable and uncontrollable. Only chance, in the form of the particular perturbation that may have prevailed at the moment of the experiment, will decide whether a given cell is right-or lefthanded. We thus arrive at a remarkable cooperation between chance and determinism . . . Stated more formally, several solutions are possible for the same parameter value. Chance alone will decide which of these solutions is realized.[53]

This line of argument for a different interpretation of the modal structure of state space is, in fact, not Deleuze's own, although it follows directly from his ontological analysis. Deleuze's own arguments against the orthodox categories of the possible and the necessary are of a more general philosophical nature,[54] and are linked directly with the third set of issues I said needed to be discussed in the remainder of this chapter: the constraints that guide Deleuze's speculation about virtuality. I have already mentioned one such constraint, to avoid at all costs conceptualizing virtual multiplicities as eternal essences. Meeting this constraint requires rejecting much of what modal logic has to say about possibility and necessity. The reason is that the postulation of possible worlds existing alongside the actual world, as Quine and other critics have often remarked, almost always implies a commitment to one or another form of essentialism.[55] And, it should be emphasized, this criticism applies not only to modal philosophers but also to those physicists who seriously believe in the existence of alternate parallel universes.

When thinking about these parallel universes, both philosophers and physicists assume the existence of *fully formed individuals* populating the different possible worlds. This immediately raises a number of questions: Can the same individual exist, slightly altered, in other worlds? Can he or she maintain this identity across many worlds, after several slight alterations have accumulated? Could we identify him or her after all these changes? It is here that essences, either general or particular, are introduced to define the identity of these individuals and to guarantee its preservation across worlds. There are basically two different technical ways of achieving this effect. On one hand, one can claim that transworld identity is insured by the possession of a *particular essence,* that is, the property of being this particular individual. On the other hand, one can deny that there are, in fact, such transworld individuals, and speak simply of *counterparts,* that is, other possible individuals which closely resemble their real counterpart, but are not identical to it (in particular, they do not share the essence of being precisely this individual). These counterparts, however, would share a general essence. (Such as being "rational animals", in the case of human beings.[56])

The alternative offered by Deleuze is to avoid taking as given fully formed individuals, or what amounts to the same thing, to always *account for the genesis of individuals* via a specific individuation process, such as the developmental process which turns an embryo into an organism. This emphasis on the objective production of the spatiotemporal structure and boundaries of individuals stands in stark contrast with the complete lack of process mediating between the possible and the real in orthodox modal thinking. The category of the possible assumes a set of predefined forms which retain their identity despite their non-existence, and which already resemble the forms they will adopt once they become realized. In other words, unlike the individuation process linking virtual multiplicities and actual structures, realizing a possibility does not add anything to the pre-existing form but mere reality. As Deleuze writes:

> What difference can there be between the existent and the non-existent if the non-existent is already possible, already included in the concept and having all the characteristics that the concept confers upon it as a possibility? . . . The possible and the virtual are . . . distinguished by the fact that one refers to the form of identity in the concept, whereas the other designates a pure multiplicity . . . which radically excludes the identical as a prior condition . . . To the extent that the possible is open to "realization" it is understood as an image of the real, while the real is supposed to resemble the possible. That is why it is difficult to understand what existence adds to the concept when all it does is double like with like . . . Actualization breaks with resemblance as a process no less than it does with identity as a principle. In this sense, actualization or differenciation is always a genuine creation. Actual terms never resemble the singularities they incarnate . . . For a potential or virtual object to be actualized is to create divergent lines which correspond to—without resembling—a virtual multiplicity.[57]

Besides the avoidance of essentialist thinking, Deleuze's speculation about virtuality is guided by the closely related constraint of avoiding *typological* thinking, that style of thought in which individuation is achieved through the *creation of classifications and of formal criteria for membership in those classifications.* Although some classifications are essentialist, that is, use transcendent essences as the criterion for membership in a class, this is not always the case. For example, unlike Platonic essences which are transcendent entities, Aristotle's "natural states", those states towards which an individual tends, and which would be achieved if there were not interfering forces, are not transcendent but *immanent* to those individuals. But while Aristotelian philosophy is indeed non-essentialist it is still completely typological, that is, concerned with defining the criteria which group individuals into species, and species into genera.[58]

For the purpose of discussing the constraints guiding Deleuze's constructive project, one historical example of typological thinking is particularly useful. This is the classificatory practices which were common in Europe in the seventeenth and eighteenth centuries, such as those that led to the botanical taxonomies of Linnaeus. Simplifying somewhat, we may say that these classifications took as a point of departure perceived *resemblances* among fully formed individuals, followed by precise comparisons aimed at an exhaustive listing of what differed and what stayed the same among those individuals. This amounted to a translation of their visible features into a linguistic representation, a tabulation of differences and *identities* which allowed the assignment of individuals to an exact place in an ordered table. Judgments of *analogy* between the classes included in the table were used to generate higher-order classes, and relations of *opposition* were established between those classes to yield dichotomies or more elaborate hierarchies of types. The resulting biological taxonomies were supposed to reconstruct a natural order which was *fixed and continuous,* regardless of the fact that historical accidents may have broken that continuity. In other words, given the fixity of the biological types, *time itself* did not play a constructive role in the generation of types, as it would later on in Darwin's theory of the evolution of species.[59]

Deleuze takes the four elements which inform these classificatory practices, *resemblance, identity, analogy* and *opposition* (or contradiction) as the four categories to be avoided in thinking about the virtual.

Deleuze, of course, would not deny that there are objects in the world which resemble one another, or that there are entities which manage to maintain their identity through time. It is just that resemblances and identities must be treated as *mere results* of deeper physical processes, and not as fundamental categories on which to base an ontology.[60] Similarly, Deleuze would not deny the validity of making judgments of analogy or of establishing relations of opposition, but he demands that we give an account of that which allows making such judgments or establishing those relations. And this account is not to be a story about us, about categories inherent in our minds or conventions inherent in our societies, but a story about the world, that is, about the objective individuation processes which yield analogous groupings and opposed properties. Let me illustrate this important point.

I said before that a plant or animal species may be viewed as defined not by an essence but by the process which produced it. I characterize the process of *speciation* in more detail in the next chapter where I also discuss in what sense a species may be said to be *an individual,* differing from organisms only in spatio-temporal scale. The individuation of species consists basically of two separate operations: a sorting operation performed by natural selection, and a consolidation operation performed by reproductive isolation, that is, by the closing of the gene pool of a species to external genetic influences. If selection pressures happen to be uniform in space and constant in time, we will tend to find more resemblance among the members of a population than if those selection forces are weak or changing. Similarly, the degree to which a species possesses a clear-cut identity will depend on the degree to which a particular reproductive community is effectively isolated. Many plant species, for example, retain their capacity to hybridize throughout their lives (they can exchange genetic materials with other plant species) and hence possess a less clear-cut genetic identity than perfectly reproductively isolated animals. In short, the degree of resemblance and identity depends on contingent historical details of the process of individuation, and is therefore not to be taken for granted. For the same reason, resemblance and identity should not be used as fundamental concepts in an ontology, but only as derivative notions.

In addition to showing, case by case, how similarity and identity are contingent on the details of an individuation process, the rejection of static categories and essences must be extended to all *natural kinds,* not just biological ones. We must show, also case by case, how terms which purport to refer to natural categories in fact refer to *historically constituted individuals.* In a way terms like "human" are the easiest to de-essentialize given that Darwin long ago gave us the means to think about species as historical entities. But what of terms like "gold" where the essentialist account seems more plausible? After all, all samples of gold must have certain atomic properties (such as having a specific atomic number) which, it can be plausibly argued, constitute the essence of gold. Part of the answer is that all atoms, not only gold atoms, need to be individuated in processes occurring within stars (nucleosynthesis), and that we can use these processes to specify what gold is instead of, say, giving its atomic number.[61] But a more compelling reason to reject essentialism here would be to deny that a given sample of gold large enough to be held in one's hand can be considered a mere sum of its atoms, hence reducible to its atomic properties.

In particular, much as between individual cells and the individual organisms which they compose there are several intermediate structures bridging the two scales (tissues, organs, organ systems) so between individual atoms of gold and an individual bulk piece of solid material there are intermediately scaled structures that bridge the micro and macro scales: individual atoms form crystals; individual crystals form small grains; individual small grains form larger grains, and so on. Both crystals and grains of different sizes are individuated following specific causal processes, and the properties of an individual bulk sample emerge from the causal interactions between these intermediate structures. There are some properties of gold, such as having a specific melting point, for example, which by definition do not belong to individual gold atoms since single atoms do not melt. Although individual gold crystals may be said to melt, in reality it takes a population of crystals with a minimum critical size (a so-called "microcluster") for the melting point of the bulk sample to emerge. Moreover, the properties of a bulk sample do not emerge all at once at a given critical scale but appear one at a time at different scales.[62]

In conclusion, avoiding essentialist and typological thinking in all realms of reality are basic requirements in the construction of a Deleuzian ontology. But besides these *negative constraints* there must be some *positive resources* which we can use in this construction. I will develop these resources in the following chapter from a more detailed analysis of the intensive processes of individuation which actualize virtual multiplicities. The virtual, in a sense, leaves behind traces of itself in the intensive processes it animates, and the philosopher's task may be seen as that of a detective who follows these tracks or connects these clues and in the process, creates a reservoir of conceptual resources to be used in completing the project which this chapter has only started. This project needs to include, besides defining multiplicities as I did above, a description of how a population of multiplicities can form a virtual continuum, that is, it needs to include a theory of *virtual space.* Similarly, if the term "virtual multiplicity" is not to be just a new label for old timeless essences, this project must include a theory of *virtual time,* and specify the relations which this non-actual temporality has with actual history. Finally, the relationship between virtuality and the *laws of physics* needs to be discussed, ideally in such a way that general laws are replaced by universal multiplicities while preserving the objective content of physical knowledge. Getting rid of laws, as well as of essences and reified categories, can then justify the introduction of the virtual as a novel dimension of reality. In other words, while introducing virtuality may seem like an inflationary ontological move, apparently burdening a realist philosophy with a complete new set of entities, when seen as a replacement for laws and essences it actually becomes deflationary, leading to an ultimately leaner ontology.

Notes

1 The term "multiplicity" makes its first appearance, as far as I can tell, in 1966 in Deleuze's book on Bergson, Gilles Deleuze, *Bergsonism* (Zone Books, New York, 1988), p. 39. Its final appearance occurs in Deleuze's last book in collaboration with Félix Guattari, Gilles Deleuze and Félix Guattari, *What Is Philosophy?* (Columbia University Press, New York, 1994), p. 15.

2 Morris Kline, *Mathematical Thought from Ancient to Modern Times,* Vol. 3 (Oxford University Press, New York, 1972), p. 882. (My emphasis)

Making surfaces into spaces, by eliminating the supplementary dimension, allowed the differentiation and study of different metric geometries. As Morris Kline writes:

> Thus if the surface of the sphere is studied as a space in itself, it has its own geometry, and even if the familiar latitude and longitude are used as the coordinates of points, the geometry of that surface is not

> Euclidian . . . However the geometry of the spherical surface is Euclidian if it is regarded as a surface in three-dimensional space. (p. 888)

For the details on Gauss coordinatization procedure, which is what guarantees this absence of a supplementary dimension or embedding space, see Lawrence Sklar, *Space, Time, and Space–Time* (University of California Press, Berkeley, 1977), pp. 27–42.

3 *Kline*, Mathematical Thought, *p. 890*.

4 Gilles Deleuze, *Difference and Repetition* (Columbia University Press, New York, 1994), p. 182. On page 183, for example, he says: "In all cases the multiplicity is intrinsically defined, without external reference or recourse to a uniform space in which it would be submerged." See also Gilles Deleuze and Félix Guattari, *A Thousand Plateaus* (University of Minnesota Press, Minneapolis, 1987), pp. 8–9,

> Unity always operates in an empty dimension supplementary to that of the system considered (overcoding) . . . [But a] multiplicity never allows itself to be overcoded, never has available a supplementary dimension over and above its number of lines, that is, over and above the multiplicity of numbers attached to those lines.

5 Deleuze and Guattari, *A Thousand Plateaus*, p. 266. The remark quoted is made about the "plane of consistency" not about multiplicities. But the former is nothing but the space formed by the multiplicities themselves, as I will explain in detail in the next chapter.

6 When Deleuze defines his multiplicities he always seems to be referring to manifolds whose dimensions are used to represent degrees of freedom (or independent variables) of some dynamic, and not to manifolds as mere geometric objects. Thus, in his first introduction of the term he says,

> Riemann defined as "multiplicities" those things that could be determined by their dimensions or their independent variables. He distinguished between discrete multiplicities and continuous multiplicities. The former contain the principle of their own metrics . . . The latter found a metrical principle in something else, even if only in phenomena unfolding in them or in the forces acting in them. *(Bergsonism,* p. 39)

And elsewhere he says, using the word "Idea" to refer to concrete universals or multiplicities as replacements for essences,

> An Idea is an n-dimensional, continuous, defined multiplicity. Colour—or rather, the Idea of colour—is a three-dimensional multiplicity. By dimensions, we mean the variables or coordinates upon which a phenomenon depends; by continuity, we mean the set of relations between changes in these variables . . . by definition, we mean the elements reciprocally determined by these relations, elements which

cannot change unless the multiplicity changes its order and its metric. (*Difference and Repetition*, p. 182)

7 I take this rather simplified description from Ian Stewart. *Does God Play Dice? The Mathematics of Chaos* (Basil Blackwell, Oxford, 1989), Chapter 6.

8 Looking for relationships between the different solution curves [i.e. trajectories] of the same differential equation, Poincaré began with a local analysis and examined the behavior of these curves in the neighborhood of a singular point . . . He showed that there were four possible different types of singular points and classified them by the behavior of the nearby solution curves: *nauds* (nodes), through which an infinite number of solution curves pass; *cols* (saddle points), through which only two solution curves pass . . . *foyers* (foci), which the solution curves approach in the manner of a logarithmic spiral; and *centres* (centers), around which the solution curves are closed, enveloping one another. Having used direct algebraic computation to show that these four types necessarily exist, he studied their distribution. He found that in the general case only three types prevailed—nodes, saddle points and foci—with centers arising in only exceptional circumstances. (June Barrow-Green, *Poincaré and the Three Body Problem* [American Mathematical Society, 1997], p. 32)

Roughly, we can say that Poincaré discovered not only the *existence* of certain recurrent "topological forms" which are bound to appear in a large class of different physical models, but also that some of these forms are "more generic" than others, that is, that if we study the *distribution* of singularities in many different models some of them (centers) are less likely to occur than others. See also discussion of the term "generic", a technical term whose meaning is still evolving, in Ralph Abraham and Christopher Shaw, *Dynamics: The Geometry of Behavior*, Vol. Three (Aerial Press, Santa Cruz, 1985), pp. 19–34.

9 Deleuze and Guattari, *A Thousand Plateaus*, p. 408.

10 "To reverse Platonism", as Deleuze says, we need "first and foremost to remove essences and to substitute events in their place, as jets of singularities" (Gilles Deleuze, *Logic of Sense* [Columbia University Press, New York, 1990], p. 53).

11 Speaking of the image of the light of reason (or of rationality as a faculty capable of grasping the essential truth of things) Deleuze says,

> The very conception of a natural light is inseparable from a certain value supposedly attached to the Idea—namely, "clarity and distinctness' . . . The restitution of the Idea in the doctrine of the faculties requires the explosion of the clear and distinct, and the discovery of a Dionysian value according to which *the Idea is necessarily obscure in so far as it is distinct,* all the more obscure the more it is distinct." (Emphasis in the original; Gilles Deleuze, *Difference and Repetition*, p. 146)

The term "Idea" here refers to multiplicities, and the fact that Deleuze uses that Platonic term shows he means to replace essences with multiplicities,

> Ideas are by no means essences. In so far as problems are the object of Ideas, problems belong on the side of events, affections, or accidents, rather than of theorematic essences . . . Consequently the domain of Ideas is that of the inessential. (p. 187)

12 Self-assembly during [the early stages of] embryonic development is not mediated by direct gene intervention. When all the transcriptions have been prevented [through the use of an inhibitor] the regular cleavage patterns are retained. However, the polarity of molecular organization of both the egg's cytoplasm and its nucleus . . . are essential for normal development. Hence the main features of [early] embryogenesis—cell differentiation, induction, determination of pattern formation—all stem from the oogenetically originated, spatial distribution of preformed informational macromolecules. The initial condition of embryogenesis is oogenesis. The epigenetics of embryonic development is built on the topological self-organization and orientation of macromolecules of the total egg. (Vladimir Glisin, *"Molecular Biology in Embryology. The Sea Urchin Embryo",* in *Self-Organizing Systems. The Emergence of Order,* ed. Eugene Yates [Plenum, New York 1987], p. 163)

The term "oogenesis" refers to the process which creates the egg in the first place.

13 Joe Rosen, *Symmetry in Science* (Springer-Verlag, New York, 1995), Chapter 2.

Besides closure, a collection of entities together with a rule of combination needs to display associativity, and possession of identity and inverse elements. The set of positive integers (including zero, and using addition as a combination rule) displays associativity because the result of adding two numbers first, and then adding a third one is the same as that of adding the first to what results from adding the last two. It also contains an "identity element", that is, an element which added to any other leaves the latter unchanged (in this case the identity element is the number zero). But it fails to be a group because it lacks inverse elements, those which when composed with certain others yield the identity element. For instance, the number "–3" when composed with the number "+3" does yield zero (which is the identity element) but "–3" is not part of the set of positive integers. Thus, for the integers to form a group we must also include negative numbers in the set.

14 This dynamic aspect of symmetry-based classifications is obscured in standard presentations of the subject by the fact that the emphasis is not placed on the transformation as an event, but on its input and output. That is, the transformation is a process but all that matters mathematically is the initial and final states of the object transformed. See Ian Stewart and Martin Golubitsky, *Fearful Symmetry* (Blackwell, Oxford, 1992), pp. 32–3.

15 *Ibid*., p. 97.

Besides assuming ideal solids and gases, this illustration of broken symmetry assumes that the gas container and the crystal lattice are infinite in all directions. The use of an "observer" to define invariance is just a convenience. The subjective point of view can, in fact, be avoided. See Joe Rosen, *Symmetry in Science*, pp. 173–4.

16 Stewart and Golubitsky, *Fearful Symmetry*, Chapter 7.

17 Ralph Abraham and Christopher Shaw, "Dynamics: A Visual Introduction", in *Self-Organizing Systems*, ed. Yates, p. 576.

18 Stewart and Golubitsky, *Fearful Symmetry,* Chapter 5. See also, Gregoire Nicolis and Ilya Prigogine, *Exploring Complexity* (W. H. Freeman, New York 1989), pp. 12–15.

19 Brian C. Goodwin, "The Evolution of Generic Forms", in *Organizational Constraints on the Dynamics of Evolution*, ed. J. Maynard Smith and G. Vida (Manchester University Press, Manchester 1990), pp. 113–14.

20 Deleuze, *Difference and Repetition*, p. 187.

Although Deleuze does not explicitly use the term "symmetry-breaking cascade", he does refer to an "embedding of groups" (p. 180) precisely in the context of explaining how a multiplicity may be progressively determined. Unfortunately, his brief discussion of groups uses a very obscure aspect of Galois's method, the originator of group theory, called the "adjunction of fields". The two formulations are, nevertheless, equivalent, fields of numbers and groups being two related nineteenth-century abstract objects. An algebraic problem, specified progressively as its field is completed by successive adjunctions, is the equivalent of an abstract smooth space being specified by a progressive series of broken symmetries, yielding increasingly more differentiated, more striated spaces. Deleuze's discussion of Galois is correct technically, but it is not as clear and intuitive as the equivalent formulation in terms of "embedding of groups". Hence in this reconstruction I will stick with the clearer alternative. But whether one uses fields or groups, it is clear that some form of *progressive differentiation* is a key component of the concept of a Deleuzian multiplicity.

21 What distinguishes a space as opposed to a mere set of points is some concept that binds the points together. Thus in Euclidean space the distance between points tells how close points are to each other . . . As Frechet [a pioneer in the development of topology] pointed out, the binding property need not be the Euclidean distance function. In particular he generalized the notion of distance by introducing the class of metric spaces. In a metric space, which can be a two-dimensional Euclidean space, one speaks of the neighborhood of a point and means all those points whose distance from the point is less than some quantity . . . However, it is also possible to suppose that the neighborhoods, certain subsets of a given set of points, are specified in some way, *even without the introduction of a metric.* Such spaces are said to have a neighborhood topology. (Morris Kline, *Mathematical Thought*, p. 1160; my emphasis)

I will use the terms "metric space" and "nonmetric space" throughout this book in the sense in which they are defined in this quote but I will take some liberties. I will speak of topological spaces, for example, as the "least metric" and of Euclidean as the "most metric", even though it would be more technically correct to differentiate *features of spaces that do or do not depend on any strictly metric property.*

22 Deleuze usually speaks (following Bergson) of two different *types* of multiplicities, metric and nonmetric, which he calls "striated" and "smooth". For the purposes of ensuring the correct interpretation of Deleuze's position here it would have been very useful if he had ever discussed Felix Klein's work, thereby clarifying the relations between the metric and the nonmetric as one of group inclusion. Unfortunately, as far as I can tell, Deleuze never discusses Klein. On the other hand, Deleuze is perfectly aware of the existence of several nonmetric geometries and uses *a single term* ("smooth space") to refer to all of them:

> It is the difference between a smooth *(vectorial, projective, or topological)* space and a striated *(metric)* space: in the first case "space is occupied without counting" and in the second case "space is counted in order to be occupied". (Deleuze and Guattari, *A Thousand Plateaus*, p. 361; my emphasis)

The definitions given in the extract are his own, but are linked to the more orthodox definitions. A metric space is counted in order to be occupied in the sense in which sedentary cultures divide the land into measured (or counted) plots in order to inhabit it:

> Good sense is . . . agricultural, inseparable from the agrarian problem, the establishment of enclosures, and the dealings of middle classes the parts of which are supposed to balance and to regulate one another. The steam engine and livestock, but also properties and classes, are the living sources of good sense, not only as facts that spring up at a particular period, but as eternal archetypes. (Deleuze, *Logic of Sense*, p. 76)

To the sedentary way of metricizing space, of dealing with it as essentially extensive, Deleuze opposes an intensive way of occupying space the way a liquid does, that is, occupying it without dividing it or counting it. This alternative he calls a "nomadic distribution". The distinction between sedentary and nomadic distributions is first made in *Difference and Repetition*, pp. 36–7, in relation to questions of typological thinking, but is taken further in an actual comparison of nomad and sedentary cultures

> . . . even though the nomadic trajectory may follow trails or customary routes, it does not fulfill the function of the sedentary road, which is to *parcel out a closed space to people,* assigning each person a share and regulating the communication between shares. The nomadic trajectory does the opposite: it *distributes people* (*or animals*) *in an open space* . . . sedentary space is striated [i.e. metricized], by walls, enclosures and roads between enclosures, while nomadic space is smooth [i.e. non-metric], marked only by "traits" that are effaced and displaced with the trajectory. (Deleuze and Guattari, *A Thousand Plateaus*, p. 380; emphasis in the original)

23 Morris Kline, *Mathematical Though*t, p. 917.

24 David A. Brannan, Matthew F. Esplen, Jeremy J. Gray, *Geometry* (Cambridge University Press, Cambridge, 1999), p. 364.

25 This way of describing the subject oversimplifies things somewhat. First of all, the actual relations between the different geometries are more complex than the simplified hierarchy "topological—differential—projective—affine—Euclidean geometries" may suggest. For the details of Klein's original classification see *ibid*., p. 919.

My friend the mathematician Andreas Dress (personal communication) summarizes Klein's programme (called the Erlanger Program) like this,

> The Erlanger Program by Felix Klein is based on the fact that depending on which (bijective) transformations you need to deal with (isometries keeping distances invariant, similarities scaling all distances by the same factor and, hence, keeping ratios of distances invariant, affine maps keeping ratios of distances of points on parallel lines invariant, projectivities keeping cross-ratios of distances invariant, differential transformations respecting infinitesimal straightness, homeomorphisms respecting nothing but infinitesimal closeness), it always makes sense to ask (1) which features of configurations within the space of interest do remain invariant, and (2) whether a basic family of such features can be found so that every other such feature can be expressed as a function of those basic ones.

26 Morris Kline, *Mathematical Thought*, p. 921. There are important exceptions to this statement. Some mathematicians, like Riemann himself, but also William Clifford, did see an ontological connection between the metric and nonmetric properties of spaces. As one historian of twentieth-century physics writes,

> [Riemann] asserted that space in itself was nothing more than a three-dimensional manifold devoid of all form: it acquired a definite form only through the material content filling it and determining its metric relations . . . Riemann's anticipation of such a dependence of the metric on physical data later provided a justification for avoiding the notion of

absolute space whose metric is independent of physical forces. For example, more than sixty years later, Einstein took Riemann's empirical conception of geometry using it as an important justification for his general theory of relativity.

(Tian Yu Cao, *Conceptual Development of Twentieth-Century Field Theories* [Cambridge University Press, Cambridge, 1997], p. 373)

27 Gordon Van Wylen, *Thermodynamics* (John Wiley & Sons, New York, 1963), p. 16.

28 What is the significance of these indivisible distances that are ceaselessly transformed and cannot be divided or transformed without their elements changing in nature each time? Is it not the intensive character of this type of multiplicity's elements and the relations between them? Exactly like a speed or a temperature, which is not composed of other speeds or temperatures, but rather is enveloped in or envelops others, each of which marks a change in nature. The metrical principle of these multiplicities is not to be found in a homogeneous milieu but resides elsewhere, in forces at work within them, in physical phenomena inhabiting them . . . (Deleuze and Guattari, *A Thousand Plateaus*, pp. 31–3)

The term "distance" is used as if it was a nonmetric property, though in its usual meaning it certainly denotes something metric. Deleuze takes this special intensive meaning of "distance" from Bertrand Russell as I will discuss in detail later in the next chapter. On distances as intensive magnitudes, or as "indivisible asymmetrical relations" see Deleuze, *Difference and Repetition,* p. 237. Deleuze does not explicitly give phase transitions as examples of "changes in kind". But one of the very few illustrations he does give is indeed a symmetry-breaking transition, "For example, one can divide movement into the gallop, trot, and walk, but in such a way that what is divided changes in nature at each moment of the division . . . " (Deleuze and Guattari, *A Thousand Plateaus*, p. 483).

On phase transitions in animal movement as broken symmetries see, Stewart and Golubitsky, *Fearful Symmetry*, Chapter 8.

29 Cao, *Conceptual Development of Twentieth-Century Field Theories*, p. 283.

30 The essential idea of grand unified theories . . . [is] the general form of hierarchical symmetry breaking: an underlying large gauge symmetry of all interactions is broken down in a succession of steps, giving a hierarchy of broken symmetries. (*ibid*., p. 328)

31 It is beyond the scope of this chapter to analyse Einstein's use of differential manifolds in technical detail. But I should at least mention the way in which his usage differs from that of Deleuze. In Einstein's theory a gravitational field constitutes the metric structure of a four-dimensional manifold (spacetime), and to this extent, the metric properties of space (rather, spacetime) are indeed connected to the physical processes which occur within it. However, as the philosopher of science Lawrence Sklar reminds us, despite the fact that Einstein's field equation does relate the metric of a manifold to the distribution of mass and energy, the relation between the two is not genetic: the metric is *not caused* by the mass—energy distribution, it is only associated with it in a lawlike way. See Sklar, *Space, Time, and Space—Time*, pp. 50–1.

32 The move away from metamathematics (set theory) and back to the actual mathematics used by scientists was initiated by the philosopher Patrick Suppes. Yet the credit for the introduction of state space into modern analytical philosophy, as well as the credit for emphasizing physical modality in the analysis of that space, goes to another philosopher, Bas Van Fraasen. See Bas Van Fraasen, *Laws and Symmetry* (Clarendon Press, Oxford, 1989), Chapter 9.

33 Ralph Abraham and Christopher Shaw, *Dynamics: The Geometry of Behavior*, Vol. 1 (Aerial Press, Santa Cruz, 1985), pp. 20–1. My description is merely a paraphrase of the following description:

> The modeling process begins with the choice of a particular state space in which to represent the system. Prolonged observations lead to many trajectories within the state space. At any point on any of these curves, a velocity vector may be derived [using the differentiation operator]. It is useful in describing an inherent tendency of the system to move with a habitual velocity, at particular points in the state space. The prescription of a velocity vector at each point in the state space is called a *velocity vector field.* The state space, filled with trajectories, is called the *phase portrait* of the dynamical system. The velocity vector field has been derived from the phase portrait by *differentiation* . . . The phrase *dynamical system* will specifically denote this vector field. (Emphasis in the original)

34 Albert Lautman, quoted in Gilles Deleuze, *Logic of Sense* (Columbia University Press, New York, 1990), p. 345. (My emphasis)

Lautman's *Le Probl??me du Temps* (from which this extract is taken) and "Essai sur le Notion de Structure et d'Existence en Mathematiques", are Deleuze's main sources on the ontological analysis of state space. Deleuze paraphrases Lautman's description in other books, but given the centrality of these ideas in his work I prefer to quote Lautman's own words.

35 Abraham and Shaw, *Dynamics: The Geometry of Behavior*, pp. 35–6.

36 Nicolis and Prigogine, *Exploring Complexity*, pp. 65–71.

37 Abraham and Shaw, *Dynamics: The Geometry of Behavior*, pp. 37–41.

38 Abraham and Shaw, *Dynamics: A Visual Introduction*, p. 562.

39 Deleuze, *Difference and Repetition*, pp. 208–9. (Emphasis in the original.) Deleuze borrows the ontological distinction of the actual and the virtual from Bergson. See Deleuze, *Bergsonism*, pp. 96–7.

40 Willard Van Orman Quine, quoted in Nicholas Rescher, "The Ontology of the Possible", in *The Possible and the Actual*, ed. Michael J. Loux (Cornell University Press, Ithaca, 1979), p. 177.

41 For a brief account of the recent history of modal logic, see Michael J. Loux, "Introduction: Modality and Metaphysics", in *Loux, The Possible and the Actual*, pp. 15–28.

42 Ronald N. Giere, "Constructive Realism", in *Images of Science. Essays on Realism and Empiricism with a Reply by Bas C. Van Fraasen*, eds. Paul M. Churchland and Clifford A. Hooker (University of Chicago Press, 1985), p. 84.

43 Bas Van Fraasen, *Laws and Symmetry.* p. 223. Van Fraasen discusses the two standard types of laws, laws of succession (which govern the evolution of trajectories, and are exemplified by Newton's laws) and laws of coexistence (which restrict position in state space, and are illustrated by Boyle's law for ideal gases).

44 Exactly matching initial conditions in the laboratory and the model is not possible, so we normally deal with *bundles of trajectories* in state space. The statistical distribution of a small population of initial states in the model is made to match that of the errors which the experimenter may have made in preparing the real system in a particular initial condition. In what follows this point will not make much difference so I stick to the simpler case of a single trajectory.

45 Giere argues that the regularities exhibited by the possible histories reveal something about the *causal regularities* in the real physical system:

> For the modal realist, the *causal* structure of the model, and thus, to some degree of approximation, of the real system, is identical with the *modal* structure. For any real system, the functional relationship among the actual values of [the degrees of freedom] are causal not because they hold among the *actual* values in *all* such real systems but because they hold for all *possible* values of *this* particular system. (*Constructive Realism,* p. 84; emphasis in the original)

See also Ronald N. Giere, *Explaining Science. A Cognitive Approach* (University of Chicago Press, 1988), Chapter 4. Giere is, in this case, wrong. State space, as I will argue in Chapter 4, provides no causal information about the modelled processes.

46 One's attitude towards modalities has a profound effect on one's whole theory of science. Actualists . . . must hold that the aim of science is to describe the actual history of the world. For [modal realists] . . . the aim is to describe the structure of physical possibility (or propensity) and necessity. The actual history is just that one possibility that happened to be realized . . . (Giere, *Constructive Realism,* p. 84)

47 Deleuze, *Logic of Sense*, p. 54.

48 Considering that Deleuze's analysis hinges on the difference between the differentiation and integration operators of the calculus, it will be necessary to remove one traditional objection to the very idea of giving an ontological dimension to these operators. This objection is that the output of the differentiation operator (instantaneous rates of change or infinitesimals) cannot be thought of as anything but mathematical fictions. Not to do so has led in the past to many sterile speculations and controversy. However, although a vector field is indeed composed of many of these instantaneous rates of change, what matters to us here are not the "instants" themselves, taken one at a time, but the *topological invariants* which those instants display collectively, that is, the singularities of the field.

49 Stephen G. Eubank and J. Doyne Farmer, "Introduction to Dynamical Systems", in *Introduction to Nonlinear Physics,* ed. Lui Lam (Springer-Verlag, New York, 1997), p. 76.

50 Abraham and Shaw, *Dynamics: The Geometry of Behavior*, pp. 7–11.

51 Attractors are indeed defined as a "limit set" with an open inset (its basin). But the word "limit" in the definition makes all the difference in the world, since it refers precisely to the tendencies of trajectories to approach the attractor in the limit. See *ibid*., p. 44.

52 "Intuitively, according to Russell, a system is deterministic exactly if its previous states determine its later states in the exact sense in which the arguments of a function determine its values. (Van Fraasen, *Laws and Symmetry*, p. 251)

See Van Fraasen's discussion of the relation between the modal category of physical necessity and deterministic laws in Chapters 3 and 4 of *Laws and Symmetry.*

53 Nicolis and Prigogine, *Exploring Complexity*, p. 14. (Emphasis in the original.)

54 For example, the way Deleuze approaches the question of necessity is by splitting the causal link: on one hand, processes of individuation are defined as sequences of causes (every effect will be the cause of yet another effect) while singularities become *pure incorporeal effects* of those series of causes; on the other hand, these pure effects are viewed as having a quasi-causal capacity to affect causal processes. By splitting causality this way, Deleuze manages to separate the determinism which links causes to causes, from strict necessity. See *Logic of Sense*, p. 169.

Deleuze uses the word "determinism" as synonymous with "necessity", and uses the word "destiny" instead for the modified link between causes. I keep the word "determinism" to avoid introducing neologisms, but emphasize the break with strict necessity. Another way of expressing Deleuze's conceptualization of this modality is from *Difference and Repetition*, p. 83,

> Destiny never consists in step-by-step deterministic relations between presents which succeed one another . . . Rather, it implies between successive presents *non-localizable connections,* actions at a distance, systems of replay, resonances and echoes . . . which transcend spatial locations and temporal successions." (My emphasis)

The idea of "non-localizable connections" is the key concept here and can be understood by reference to convection cells. While the causal interactions between the cell's components are localizable collisions (billiard-ball style causality), the source of coherence in the flow pattern (the periodic attractor) is, indeed, nowhere specifically in space or time. The attractor establishes connections (else there would be no coherence in the flow) but not localizable ones.

55 Willard Van Orman Quine, "Reference and Modality", in *From a Logical Point of View* (Harper & Row, New York, 1965), p. 155. Even though most modal analyses deal with purely linguistic phenomena, such as counterfactual sentences, the moment one approaches such sentences as referring to the real world (technically, the moment we quantify over possible entities) we acquire an ontological commitment to the existence of essences. In other words, we commit ourselves to affirm that objects possess some of their properties necessarily while others only contingently.

56 The first option (ensuring transworld identity through particular essences or hacceities) is exemplified by Alvin Plantinga, "Transworld Identity or World-bound Individuals?", in *Loux, The Possible and the Actual*, pp. 154–7.

The second option (counterparts linked through general essences) is illustrated by David Lewis, "Counterpart Theory and Quantified Modal Logic", in *The Possible and the Actual*, pp. 117–21.

57 Deleuze, *Difference and Repetition*, pp. 211–12. See also Deleuze, *Bergsonism,* p. 97. Deleuze does not, in fact, refer to the virtual as a physical modality, but the fact that he explicitly contrasts virtuality and possibility (following Bergson's lead) does indicate that he is thinking in modal terms.

58 I take this description of Aristotelian philosophy from Elliot Sober, *The Nature of Selection* (MIT Press, Cambridge, 1987), pp. 156–61.

59 Deleuze, *Difference and Repetition,* p. 29. To avoid falling prey to the dangers of representationalism (or as I call it typological thinking) Deleuze follows Michel Foucault's analysis of classical representation, which according to the latter forms an epistemological space with four dimensions or "degrees of freedom": identity, resemblance, analogy and opposition, p. 262.

For a discussion of this aspect of Foucault's thought from the point of view of an analytical philosopher see Gary Gutting, *Michel Foucault's Archaeology of Scientific Reason* (Cambridge University Press, 1993), Chapter 4.

In what follows I simply take the idea that there are recurrent features in these classificatory practices (resemblance, identity, etc.) but not that these form a global entity called an "episteme". I do not believe such global entities or totalities exist as will become clear in the following chapters.

60 "The first formula posits resemblance as the condition of difference. It therefore undoubtedly demands the possibility of an identical concept for the two things that differ on condition that they are alike . . . According to the other formula, by contrast, resemblance, identity, analogy and opposition can no longer be considered anything but effects of a primary difference or a primary system of differences. (Deleuze, *Difference and Repetition*, p. 117)

Deleuze, in fact, does not speak of "constraints guiding a constructive project". He rather affirms his desire for creating a *philosophy of difference,* and then denounces the categories of typological or representational thinking as obstacles to reaching that goal. The differences he has in mind are not the external *differences between things* that are part and parcel of classificatory practices, but productive differences perhaps best illustrated by *intensive differences,* differences in temperature, pressure, etc. within one and the same system, which are marked by thresholds of intensity determining phase transitions. See p. 222.

61 Ronald F. Fox, *Energy and the Evolution of Life* (W. H. Freeman, New York, 1988), p. 8.

> The mechanisms by which the chemical elements come into existence is *stellar nucleosynthesis.* The processes involved are an example of how *energy flow* produces complex states of matter from simpler constituents. A combination of gravitational energy and nuclear energy converts vast quantities of hydrogen gas, the simplest element, into the nuclei of other more complex elements. Nucleosynthesis involves nuclear reaction cycles and happens in stages that correlate strongly with changes in stellar structure. (Emphasis in the original)

62 Philosophers tend to imagine that a piece of bulk material is simply a collection of individual crystals arranged so perfectly that, for all practical purposes, the properties of the bulk sample are simply a sum of the properties of these crystals. In other words, they imagine we can *divide the bulk sample in extension* and, given the packing arrangement of the crystals, we will always end up with a similar if smaller sample. But in reality, we do not have perfectly regular crystal lattices (the irregularities playing a crucial role in the stability of the structure) and we cannot divide a bulk sample beyond a given size without losing some emergent properties:

> Like the biologist, the metallurgist is concerned with aggregates and assemblies in which repeated or extended *irregularities* in the arranged atoms become the basis of major structural features on a larger scale, eventually bridging the gap between the atom and things perceptible to human senses. (Cyril Stanley Smith, "Structure, Substructure, and Superstructure", in *A Search for Structure* [MIT Press, Cambridge, 1982], p. 54; my emphasis)

See also, in the same volume, Smith, "Grain Shapes and other Metallurgical Applications of Topology". On the emergence of bulk properties at different critical scales, see Michael A. Duncan and Dennis H. Rouvray, *Microclusters* (Scientific American, December, 1989), p. 113.